Modelling of Periodically Switching Networks

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ABSTRACT

This paper offers two new time-domain analysis methods for periodically switched systems, like converters. The first method analyses the steady-state periodic behaviour of the system, while the second describes the transient behaviour. In contrast with state-space averaging the methods do not use approximations, so exact results are obtained. Moreover, the methods gives more insight into the dynamical behaviour of the system. This knowledge is useful in designing controllers.

keywords: switching networks, modelling, analysis, Floquet, state-space averaging

INTRODUCTION

One of the topics of research in modelling and control of electrical drives is the modelling of electrical converters. From a system theoretic point of view converters belong in the class of switching systems: they are characterized by the fact that their operation is based on switches. The system can be in several modes depending on the actual state of the switches. In each mode the system displays a specific dynamic behaviour, which can be described by a linear model. The state-vector of these models consists of the physical quantities of the system (like inductor currents and capacitor voltages) and is therefore the same for all models. The parameters of the models, however, depend on the state of the switches, which determine the interaction between the physical quantities. If the system is operating in a specific operating point, the switching instants are considered to be known.

We consider systems which are described by piece-wise linear models with discontinuously changing parameters. Each set of parameters corresponds to a certain *mode* of the system determined by the actual state of the switches in the system. If the system contains p switches, the system can be in at most 2^p different *modes* depending on the state of the switches. Due to physical restrictions, not all these modes are possible. Therefore, we state that there are m possible combinations of switches, and consequently the system has m modes of operation. Each mode is described by a linear model $\{A_i, b_i\}$:

$$\dot{x}(t) \quad = \quad A_i x(t) + b_i \tag{1}$$

in which $x(t)$ denotes the state vector ($\in R^n$), A_i ($n \times n$) represents the dynamics of the system and b_i ($n \times 1$) is an excitation term in which e.g. sources can be modelled. The mode of the system is given by i, with $i \in 1 \ldots m$ (m possible modes).

Further, the system is assumed to have one basic operating frequency with period T. Within a period, the modes are consecutive, each having a time interval Δt_i or in terms of the total period T

$$\Delta t_i = d_i T \qquad\qquad i \in 1 \ldots m \tag{2}$$

The time interval Δt_i is a fraction d_i (duty-cycle) of the total period T. The sum of all time intervals Δt_i equals the whole period T and $\sum_{i=1}^{m} d_i = 1$. These definitions are elucidated in figure 1.

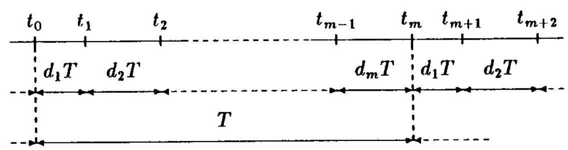

Figure 1: *Definition time intervals*

A well-known method to analyse this type of systems is the method of state-space averaging [1],[2]. State-space averaging is appealing because of its mathematical simplicity, but gives only satisfactory results if the operating frequency is high in comparison with the dynamics of the system. Also in other analysis approaches this assumption is made, e.g. [3]. If this assumption is not valid, exact modelling is required, which is usually very cumbersome. This situation arises for instance in high performance applications where volume and mass are dominant design criteria. We will introduce a new exact method for calculating important system properies like the average system behaviour and harmonics directly from the system equations. This approach is both elegant and simple and can replace exhaustive simulations.

The outline of the paper is as follows. First, some basic results from state-space averaging are summarized. Then, the proposed method is presented. Further, expressions for the steady-state behaviour, average state-vector, harmonic components of the signals and transient behaviour of the system are derived. Finally, a comparison is made and some conclusions are drawn.

STATE-SPACE AVERAGING

An important tool in analysing switched networks is the method of state-space averaging [1],[2]. It is used to analyse systems described by a series of piecewise linear models (1) with constant duty-cycles (2), as introduced in the previous section.

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The method of state-space averaging approximates the average system behaviour by the averaged dynamics $\{A_{av}, b_{av}\}$ of the system:

$$A_{av} = \sum_1^m d_i A_i \tag{3}$$

$$b_{av} = \sum_1^m d_i b_i \tag{4}$$

$$\dot{x}_{av}(t) = A_{av} x_{av}(t) + b_{av} \tag{5}$$

These expressions are valid if the following approximation holds:

$$e^{A_m d_m T} e^{A_{m-1} d_{m-1} T} \cdots e^{A_1 d_1 T} \approx e^{A_1 d_1 T + \cdots + A_m d_m T} \tag{6}$$

The difference between exact value and approximation can be neglected if T is small in comparison with the dynamics of the system as defined by the eigenvalues of A_i, $i = 1, \ldots, m$.

Expression (5) gives the evolution of the averaged state-vector from any initial condition $x(t_0)$ to the steady-state value. Thus, it describes the transient behaviour of the system. From (5), the steady-state value $x_{av,ss}$ of the averaged state-vector x_{av} can be calculated by stating that $\dot{x}_{av}(t) = 0$:

$$x_{av,ss} = -(A_{av})^{-1} b_{av} \tag{7}$$

In the next sections exact expressions for these properties will be derived.

EXACT ANALYSIS IN CYCLIC MODE

In this paragraph the behaviour of the system in cyclic mode will be analysed. In classical (stable) dynamical systems the state-vector converges to a specific steady-state value if the inputs of the system are constant. This never occurs in switching networks. Due to the constant changes in system parameters, the state-vector of the system will not converge to one specific steady-state value. The closest analogy to the notion of steady-state in this kind of systems is the notion of *cyclic mode*. A system is said to be in cyclic mode if the state-vector is periodic with the basic operating frequency of the system. Also the concept of system inputs must be reviewed. From a control engineering point of view, the inputs of a system are those variables that can actively be controlled. In the case of switching networks the control elements are the switches. Thus, the time-intervals the switches are opened or closed, as determined by the duty-cycles d_i, are the control inputs of the system. Analysing the steady-state behaviour of a switching network therefore means analysing the cyclic mode of the system, assuming constant duty-cycles for each individual mode. The problem is to find an expression for the periodic behaviour of the system given by (1) and (2).

The first step to solving this problem is finding expressions for the evolution of the state-vector $x(t)$. For $t_{i-1} \le t \le t_i$ the solution of (1) is given by

$$x(t) = e^{A_i(t-t_{i-1})} x(t_{i-1}) + \int_{t_{i-1}}^{t} e^{A_i(t-\tau)} b_i \, d\tau \tag{8}$$

So, for every mode i, the state-vector $x(t)$ at $t = t_i$ is calculated from the state at t_{i-1}:

$$x(t_i) = e^{A_i d_i T} x(t_{i-1}) + \int_{t_{i-1}}^{t_i} e^{A_i(t_i-\tau)} b_i \, d\tau \tag{9}$$

For simplicity the following notations are introduced:

$$\begin{aligned} \Phi_i &= \Phi(t_i, t_{i-1}) && (10)\\ &= e^{A_i d_i T} && (11)\\ \Gamma_i &= \int_{t_{i-1}}^{t_i} e^{A_i(t_i-\tau)} b_i \, d\tau && (12)\\ &= \left(e^{A_i d_i T} - I\right) A_i^{-1} b_i \qquad \text{(if } A_i \text{ is invertible)} && (13) \end{aligned}$$

With these expressions, the state-vector at t_i is calculated recursively from the initial condition $x(t_0)$.

$$\begin{aligned} x(t_i) &= \Phi_i x(t_{i-1}) + \Gamma_i && (14)\\ &= \Phi_i \left(\Phi_{i-1} x(t_{i-2}) + \Gamma_{i-1}\right) + \Gamma_i && (15)\\ &= \Phi_i \Phi_{i-1} x(t_{i-2}) + \Phi_i \Gamma_{i-1} + \Gamma_i && (16)\\ &= \Phi_i \Phi_{i-1} \cdots \Phi_1 x(t_0) + \Phi_i \Phi_{i-1} \cdots \Phi_2 \Gamma_1 + \\ &\quad + \cdots + \Phi_i \Gamma_{i-1} + \Gamma_i && (17) \end{aligned}$$

The periodic solution $x_{per}(t)$ of the system is found by stating that the state-vector $x_{per}(t+T) = x_{per}(t)$. For $t = t_0$ this yields that $x(t_m) = x(t_0+T) = x(t_0)$.

$$\begin{aligned} x(t_m) &= \Phi_m \Phi_{m-1} \cdots \Phi_1 x(t_0) + \Phi_m \Phi_{m-1} \cdots \Phi_2 \Gamma_1 + \\ &\quad + \cdots + \Phi_m \Gamma_{m-1} + \Gamma_m && (18) \end{aligned}$$

So,

$$\begin{aligned} x_{per}(t_0) &= (I - \Phi_m \Phi_{m-1} \cdots \Phi_1)^{-1} (\Phi_m \Phi_{m-1} \cdots \Phi_2 \Gamma_1 + \\ &\quad + \cdots + \Phi_m \Gamma_{m-1} + \Gamma_m) && (19) \end{aligned}$$

This expression can be calculated, but not easily. It contains many terms that have to be calculated separately. Also, some of the Γ_i-terms may be hard to calculate, especially if the corresponding system matrix A_i is singular.

Therefore, the augmented state-vector method is introduced. A new, augmented state-vector $\hat{x}(t)$ ($\in R^{n+1}$) and a new system matrix $\hat{A}$ ($n+1 \times n+1$) are defined, both containing dummy elements. This generates the following augmented system equation for each mode i:

$$\frac{d}{dt} \left(\begin{array}{c} \vdots \\ x(t) \\ \vdots \\ \hline 1 \end{array} \right) = \left(\begin{array}{ccc|c} & A_i & & \begin{matrix} \vdots \\ b_i \\ \vdots \end{matrix} \\ \hline 0 & \cdots & 0 & 0 \end{array} \right) \left(\begin{array}{c} \vdots \\ x(t) \\ \vdots \\ \hline 1 \end{array} \right) \tag{20}$$

$$\frac{d}{dt} \hat{x}(t) = \hat{A}_i \hat{x}(t) \tag{21}$$

The 'dummy' 1 in the state-vector ensures that the original system equation is incorporated in the new one. To make sure that the augmented dummy state remains 1, a row of zeros has been added to the system matrix $\hat{A}_i$.

Just as in the original system, the new state-vector can be calculated recursively from the initial state. The resulting expressions, however, are much simpler, because the new system is homogeneous. For each mode i the evolution of the state is given by (22,23), while the transition over a complete period T is given by (24,25).

$$\begin{aligned} \hat{x}(t_i) &= e^{\hat{A}_i d_i T} \hat{x}(t_{i-1}) && (22)\\ &= \hat{\Phi}_i \hat{x}(t_{i-1}) && (23) \end{aligned}$$

$$\begin{aligned} \hat{x}(t_m) &= \hat{\Phi}_m \hat{\Phi}_{m-1} \cdots \hat{\Phi}_1 \hat{x}(t_0) && (24)\\ &= \hat{\Phi}_{tot} \hat{x}(t_0) && (25) \end{aligned}$$

It is interesting to note that the structure of the transition matrices $\hat{\Phi}_i$ incorporates the original matrices Φ_i and Γ_i:

$$\hat{\Phi}_i = \left(\begin{array}{c|c} \Phi_i & \Gamma_i \\ \hline 0 & 1 \end{array}\right) \tag{26}$$

The periodic solution of $\hat{x}_{per}(t)$ is again found by stating that the state-vector after a complete period T must equal the initial state, thus

$$\hat{x}_{per}(t_0+T) = \hat{\Phi}_{tot}\hat{x}_{per}(t_0) = \hat{x}_{per}(t_0) \tag{27}$$

$$0 = (I^{n+1} - \hat{\Phi}_{tot})\hat{x}_{per}(t_0) \tag{28}$$

$$0 = \left(\begin{array}{c|c} I^n - \Phi_{tot} & -\Gamma_{tot} \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c} x_{per}(t_0) \\ \hline 1 \end{array}\right) \tag{29}$$

So,

$$x_{per}(t_0) = (I^n - \Phi_{tot})^{-1}\Gamma_{tot} \tag{30}$$

Using (30), (23) and (8), the cyclic state $x_{per}(t)$ can be calculated analytically for any t.

Average State

As shown it is possible to derive an analytical expression for the cyclic state of the system. This expression is nice, but unpractical from a control engineering point of view. The reason is that no clear outputs of the system have yet been defined, nor any control objective. The expressions for the cyclic state, elegant as they may be, are not a goal in itself. In mathematical terms the set of system equations and inputs (duty-cycles) is mapped onto a set of periodic functions, which gives the system an infinite number of possible outputs: the state-vector on each instant in the interval T is unique! So, before any control action is taken, the actual *objectives* of the system must be clearly defined.

Most converters are constructed to synthesize certain voltage or current forms. Then, the objectives may be formulated as reaching certain average values, with a maximum on the ripple of certain variables or certain demands on the harmonics. These values constitute the control objectives and are thus the outputs of the system that need to be controlled. Obviously, they are in close connection with the cyclic state-vector of the system, but they can *not* directly be identified as physical states of the system, rather they are functions of the cyclic state.

A useful characterization of a periodic function is its average value. Given the expressions for the cyclic state, the average state x_{av} will be calculated. This can be done in several ways. All methods satisfy the definition of average value:

$$x_{av} = \frac{1}{T}\int_{t_0}^{t_0+T} x_{per}(t)dt \tag{31}$$

Substitution of (8)-(19) in this definition is possible, but yields awkward looking expressions like

$$\begin{aligned} x_{av} = \frac{1}{T}\sum_{i=1}^{n}\Big\{ & (\Phi_i - I)\, A_i^{-1}\,(I - \Phi_{i-1,1}\Phi_{n,i})^{-1}\cdot \\ & \cdot\Big(\sum_{k=1}^{i-1} \Phi_{i-1,k+1}\, r_k + \sum_{k=i}^{n} \Phi_{i-1,1}\Phi_{n,k+1}\, r_k\Big) + \\ & + (\Phi_i - I)\, A_i^{-2} b_i - A_i^{-1} b_i d_i T\Big\} \end{aligned} \tag{32}$$

which are difficult to calculate, even with good mathematical software like MATLAB. Here also, the trick of state-vector augmentation can be applied.

Consider the following system:

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & A_i x(t) + b_i \\ \dot{y}(t) & = & \frac{1}{T}x(t) \end{array}\right. \tag{33}$$

Subsequently, define a new state-vector $z(t)$, incorporating x and y, with $z(t) = \left(\begin{array}{ccc} x^T(t) & 1 & y^T(t) \end{array}\right)^T \in R^{2n+1}$. System (33) can then be written as

$$\frac{d}{dt}\left(\begin{array}{c} x(t) \\ \hline 1 \\ \hline y(t) \end{array}\right) = \left(\begin{array}{c|c|c} A_i & b_i & 0 \\ \hline 0 & 0 & 0 \\ \hline I/T & 0 & 0 \end{array}\right)\left(\begin{array}{c} x(t) \\ \hline 1 \\ \hline y(t) \end{array}\right) \tag{34}$$

$$\dot{z}(t) = \tilde{A}_i z(t) \tag{35}$$

The matrices $\tilde{A}_i$ have dimensions $(2n+1) \times (2n+1)$. The first part of the new state-vector is nothing but the original system. The second part consists of the new state variables y, and is a direct translation of the integral (31). To calculate the average value of the periodic behaviour of the original system (1), the integral of $x_{per}(t)$ from t_0 to t_0+T is to be computed. This is done by calculating $z(t_0+T)$ from $z(t_0)$. The initial state $z(t_0)$ is chosen as

$$z(t_0) = \left(\begin{array}{c} x_{per}(t_0) \\ \hline 1 \\ \hline 0 \end{array}\right) \tag{36}$$

Then, the x part of z will display the periodic behaviour of the original system, while the y part yields the integral value of x divided by T over the whole period, which is by definition the average value:

$$z(t_0+T) = \tilde{\Phi}_m\tilde{\Phi}_{m-1}\cdots\tilde{\Phi}_1 z(t_0) \tag{37}$$

$$= \left(\begin{array}{c} x_{per}(t_0) \\ \hline 1 \\ \hline x_{av} \end{array}\right) \tag{38}$$

$$\tilde{\Phi}_i = e^{\tilde{A}_i d_i T} \tag{39}$$

This calculation method is much easier than the direct method as illustrated in (32). It incorporates only m matrix exponentials and m matrix multiplications (given $x(t_0)$) (which any numerical software package can perform) in contrast to the many different matrix summations and products of (32).

Remark 1: In this section the average of the whole state-vector was computed. If only the average value of one or a few elements is required, or perhaps a linear combination, then system (33) can be reduced accordingly. If the average of the vector $q(t) = Qx(t)$ ($q \in R^l$; $l \leq n$; $Q = l \times n$) is of interest, then the matrix I/T in (34) can be replaced by Q/T. The dimensions of $\tilde{A}_i$ and $\tilde{\Phi}_i$ reduce accordingly to $(n+l+1 \times n+l+1)$.

Remark 2: The exact value of the average state x_{av}, as calculated via (36)-(38), is a function of $\{A_i, b_i, d_i, T\}$. In contrast, the approximated average value $x_{av,ss}$, calculated via (7), only depends on $\{A_i, b_i, d_i\}$ and *not* on T.

Harmonics of Cyclic State

Another important characteristic of the cyclic state is the waveform of the different elements of the state-vector. These waveforms can be described by the Fourier components of the cyclic state. Any periodic function can be decomposed into an infinite series of orthogonal functions. These functions are usually sines and cosines or complex exponentials. Here, the periodic state-vector $x_{per}(t)$ will be decomposed into complex exponential functions for reasons of notational simplicity (40). The coefficients for the sine/cosine decomposition are, however, closely related to the ones found here (see appendix A).

$$x_{per}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \qquad \omega = \frac{2\pi}{T} \tag{40}$$

The Fourier coefficient vector c_k can be calculated from (41).

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x_{per}(t)e^{-jk\omega t}dt \tag{41}$$

In (42) the integral (41) is split into separate integrals for each mode. Also, the state-vector $x(t)$ is replaced by the augmented state-vector $\hat{x}(t)$ in the same way as in (20). This allows substitution of an explicit expression for the state-vector at t. As a consequence the vector of Fourier coefficients c_k has also been augmented with a dummy element. This element will be the Fourier coefficient of the dummy element in the state-vector and can thus be ignored.

$$\hat{c}_k = \frac{1}{T}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i} e^{\hat{A}_i(t-t_{i-1})}\hat{x}_{per}(t_{i-1})e^{-jk\omega t}dt \tag{42}$$

The complex exponential is divided into two parts and integrated with the matrix exponential. The result of these manipulations is given in (43).

$$\begin{aligned}\hat{c}_k &= \sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{1}{T}e^{-jk\omega t_{i-1}}e^{\hat{A}_i(t-t_{i-1})}e^{-jk\omega(t-t_{i-1})}\hat{x}_{per}(t_{i-1})dt \\ &= \underbrace{\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{1}{T}e^{-jk\omega t_{i-1}}\underbrace{e^{(\hat{A}_i-jk\omega I)(t-t_{i-1})}\hat{x}_{per}(t_{i-1})}_{v_k(t)}\,dt}_{w_k(t_i)}\end{aligned} \tag{43}$$

The underbraced term marked $v_k(t)$ is the solution of the system (44) with initial condition (45). The underbraced term marked $w_k(t_i)$ is the solution at t_i of the system (46) with initial condition (47). Note that both $v_k(t)$ and $w_k(t)$ are complex-valued functions.

$$\begin{aligned}\dot{v}_k(t) &= (\hat{A}_i - jk\omega I)v_k(t) &(44)\\ v_k(t_{i-1}) &= \hat{x}_{per}(t_{i-1}) &(45)\\ \dot{w}_k(t) &= \frac{1}{T}e^{-jk\omega t_{i-1}}v_k(t) &(46)\\ w_k(t_{i-1}) &= 0 &(47)\end{aligned}$$

A new fictitious system $\{\mathcal{A}_{i,k}, \varsigma_k\}$, with state-vector $\varsigma_k(t) = \begin{pmatrix} v_k^T(t) & w_k^T(t)\end{pmatrix}^T$ is introduced:

$$\begin{aligned}\frac{d}{dt}\left(\begin{array}{c} v_k(t) \\ \hline w_k(t)\end{array}\right) &= \left(\begin{array}{c|c} \hat{A}_i - jk\omega I & 0 \\ \hline \frac{1}{T}e^{-jk\omega t_{i-1}}I & 0\end{array}\right)\left(\begin{array}{c} v_k(t) \\ \hline w_k(t)\end{array}\right) &(48)\\ \dot{\varsigma}_k(t) &= \mathcal{A}_{i,k}\varsigma_k(t) &(49)\end{aligned}$$

The vector $w_k(t_i)$ is found from

$$\left(\begin{array}{c} v_k(t_i) \\ \hline w_k(t_i)\end{array}\right) = e^{\mathcal{A}_{i,k}d_iT}\left(\begin{array}{c} \hat{x}_{per}(t_{i-1}) \\ \hline 0\end{array}\right) \tag{50}$$

From (43) follows that the Fourier coefficients vector $\hat{c}_k$ is the sum of all $w_k(t_i)$

$$\hat{c}_k = \sum_{i=1}^{m} w_k(t_i) \tag{51}$$

Note that for $k = 0$ system (44)-(47) is identical to (33).

Remark: If the Fourier coefficients of some elements of the state-vector are not required, the system may be reduced as described in the remark at the end of the section about the average state. Here, the matrix $\frac{1}{T}e^{-jk\omega t_{i-1}}I$ can be replaced by $\frac{1}{T}e^{-jk\omega t_{i-1}}Q$, if the coefficients of the vector $q(t) = Qx(t)$ are of interest.

Minimum and Maximum Value

Determination of the minimum and maximum values of the state variables is a little more difficult. In general, extreme values of a function are obtained by stating that the derivative of the function equals 0. This is impossible in switching networks, since the state-vector $x_{per}(t)$ has a *discontinuous* derivative. Therefore each of the time intervals d_iT must be considered individually. For the maximum and minimum value of the state variable $x_{per}^k(t)$ (k-th element of $x_{per}(t)$) the following holds:

$$\begin{aligned}\min_t x_{per}^k(t) &= \min\{\underline{x}_i^k | i \in 1,\ldots,m\} &(52)\\ \max_t x_{per}^k(t) &= \max\{\bar{x}_i^k | i \in 1,\ldots,m\} &(53)\end{aligned}$$

with

$$\begin{aligned}\underline{x}_i^k &= \min\{x_{per}^k(t) | t \in [t_{i-1}, t_i]\} &(54)\\ \bar{x}_i^k &= \max\{x_{per}^k(t) | t \in [t_{i-1}, t_i]\} &(55)\end{aligned}$$

The problem is to find the extreme values (54) and (55). There are two possibilities: either the extreme value lies *within* the interval or the extreme value lies on the *boundary*. Within the interval the derivative of the state-variables is continuous. An extreme value therefore implies a zero-derivative and gives an equation from which the extreme could be solved. Unfortunately, this equation is not analytically solvable, so the solution can only be obtained numerically.

In most cases, however, the assumption that T is small compared to the system dynamics holds. As a consequence, the system variables tend to behave linearly with respect to time within an interval. Then, the extreme value is found on one of the boundaries of the interval. Finding the extreme values for the complete period T in that case requires examining the state-vector $x_{per}(t_i)$ only at the switching instants.

TRANSIENT TO CYCLIC MODE

In the previous section the system behaviour in cyclic mode was discussed. The next step is describing the transient behaviour of the system. For this the Floquet theorem will be used. First the Floquet theorem will be introduced for homogeneous systems. Since this result is unsatisfactory for practical systems, the theorem will be extended to non-homogeneous systems.

Basic Floquet theorem

The Floquet theorem [4] offers the possibility of exact modelling of systems with periodic coefficients. The theorem can be applied to systems described by a differential equation of the following type:

$$\dot{x}(t) = A(t)x(t) \tag{56}$$

In this equation the function $A(t)$ is a function of time. In general, the solution of this equation can be denoted as

$$x(t) = \Phi(t,t_0)x(t_0) \tag{57}$$

If $A(t)$ is a periodic function, i.e. $A(t) = A(t+T)$, the following definitions yield interesting results:

Let K be defined by

$$e^{KT} = \Phi(t_0+T,t_0) \tag{58}$$

Consequently, $K = \frac{1}{T}\log\Phi(t_0+T,t_0)$. Subsequently, let $F(t,t_0)$ be defined by

$$F(t,t_0) = \Phi(t,t_0)e^{-K(t-t_0)} \tag{59}$$

Thus $\Phi(t,t_0) = F(t,t_0)e^{K(t-t_0)}$.

The Floquet theorem says that the matrix function $F(t,t_0)$ is *periodic* with period T, so $F(t+T,t_0) = F(t,t_0)$.
Proof:

$$\begin{aligned}
F(t+T,t_0) &= \Phi(t+T,t_0)e^{-K(t+T-t_0)} \\
&= \underbrace{\Phi(t+T,t_0+T)}_{\Phi(t,t_0)}\underbrace{\Phi(t_0+T,t_0)e^{-KT}}_{I}\,e^{-K(t-t_0)} \\
&= \Phi(t,t_0)e^{-K(t-t_0)} \\
&= F(t,t_0)
\end{aligned}$$

$\Phi(t,t_0) = \Phi(t+T,t_0+T)$ since $A(t) = A(t+T)$. With these definitions the state-vector $x(t)$ can be denoted as

$$\begin{aligned}
x(t) &= \Phi(t,t_0)x(t_0) \\
&= \Phi(t,t_0)e^{-K(t-t_0)}e^{K(t-t_0)}x(t_0) \\
x(t) &= F(t,t_0)e^{K(t-t_0)}x(t_0)
\end{aligned} \tag{60}$$

This result shows that the expression for the state-vector of a system represented by a set of homogeneous periodic differential equations (56) consists of three parts:

- $F(t,t_0)$, a periodic function with period T. This function represents the periodic part $A(t) = A(t+T)$ of the system.
- $e^{K(t-t_0)}$, an exponentially damping function. This function represents the average dynamic behaviour of the system.
- $x(t_0)$, the initial state-vector of the system.

Extended Floquet theorem

The results of the standard Floquet theorem are not directly applicable to switching systems of type (1). The differential equations of the switching system considered in this paper are non-homogeneous, so they incorporate some exitation terms b_i in their description. This is not the case in standard Floquet theory. Therefore, we have extended the theorem to cope also with non-homogeneous systems. Consider the following non-homogeneous, periodic differential equation:

$$\dot{x}(t) = A(t)x(t) + b(t) \tag{61}$$

with $A(t) = A(t+T)$ and $b(t) = b(t+T)$. The solution of (61) is given by

$$x(t) = \Phi(t,t_0)x(t_0) + \Phi(t,t_0)\int_{t_0}^{t}\Phi^{-1}(\tau,t_0)b(\tau)d\tau \tag{62}$$

The periodic solution $x_{per}(t)$ of (61) is computed from (62) by stating $x_{per}(t_0+T) = x_{per}(t_0)$. This yields

$$x_{per}(t_0) = \left(I - \Phi(t_0+T,t_0)\right)^{-1}\Phi(t_0+T,t_0)\cdot \cdot\int_{t_0}^{t_0+T}\Phi^{-1}(\tau,t_0)b(\tau)d\tau \tag{63}$$

$$x_{per}(t) = \Phi(t,t_0)\left(x_{per}(t_0) + \int_{t_0}^{t}\Phi^{-1}(\tau,t_0)b(\tau)d\tau\right) \tag{64}$$

Using the Floquet definitions (58) and (59) the following expressions for the state-vector are derived:

$$\begin{aligned}
x(t) &= \Phi(t,t_0)\left(x(t_0) + \int_{t_0}^{t}\Phi^{-1}(\tau,t_0)b(\tau)d\tau\right) \\
&= \Phi(t,t_0)x(t_0) - \Phi(t,t_0)x_{per}(t_0) + \\
&\quad \underbrace{+\Phi(t,t_0)x_{per}(t_0) + \Phi(t,t_0)\int_{t_0}^{t}\Phi^{-1}(\tau,t_0)b(\tau)d\tau}_{x_{per}(t)} \\
x(t) &= \underbrace{F(t,t_0)e^{K(t-t_0)}(x(t_0) - x_{per}(t_0))}_{\to\,0,\ \text{for } t\to\infty} + x_{per}(t)
\end{aligned} \tag{65}$$

This result shows that the state-vector of a system described by (61) consists of two parts:

- $F(t,t_0)e^{K(t-t_0)}(x(t_0) - x_{per}(t_0))$, a damping term describing the transient behaviour from a generalized initial condition $(x(t_0) - x_{per}(t_0))$ to zero.
- $x_{per}(t)$, a periodic function with period T. This term represents the periodic behaviour of the system in cyclic mode.

So, for $t \to \infty$ $x(t)$ approaches $x_{per}(t)$, the state-vector in cyclic mode. Because $F(t,t_0)$ is also a periodic function, the dynamics of the transient are fully determined by $e^{K(t-t_0)}$: the eigenvalues of the matrix K determine the transient behaviour!

EXAMPLE

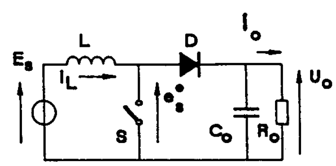

Figure 2: *Boost converter*

In this paragraph the methods outlined in the previous sections will be illustrated. As an example we have chosen the boost converter (figure 2). If the switches are considered ideal, the converter is described by the following equations:

$$x(t) = \begin{pmatrix} u_C \\ i_L \end{pmatrix} \tag{66}$$

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{pmatrix}x(t) + \begin{pmatrix} 0 \\ \frac{E}{L} \end{pmatrix} \quad \text{switch open} \tag{67}$$

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{RC} & 0 \\ 0 & 0 \end{pmatrix}x(t) + \begin{pmatrix} 0 \\ \frac{E}{L} \end{pmatrix} \quad \text{switch closed} \tag{68}$$

For this example we take $R = L = C = 1$. The period $T = 2$ s and the duty-cycles $d_1 = d_2 = 0.5$. The source voltage is 20 V.

Then the matrices $\hat{\Phi}_1$ and $\hat{\Phi}_2$ are calculated from (22):

$$\hat{\Phi}_1 = \begin{pmatrix} 0.1262 & 0.5335 & 6.8060 \\ -0.5335 & 0.6597 & 17.4761 \\ 0 & 0 & 1.0000 \end{pmatrix} \quad (69)$$

$$\hat{\Phi}_2 = \begin{pmatrix} 0.3679 & 0 & 0 \\ 0 & 1.0000 & 20.0000 \\ 0 & 0 & 1.0000 \end{pmatrix} \quad (70)$$

Using (30) and (23) the transition points of the periodic solution of $x(t)$ are found:

$$x(t_0) = (\ 19.1219 \quad 80.1483\ )^T \quad (71)$$

$$x(t_1) = (\ 51.9788 \quad 60.1483\ )^T \quad (72)$$

Application of (38) with

$$\tilde{\Phi}_{tot} = \begin{pmatrix} 0.0464 & 0.1963 & 2.5038 & 0 & 0 \\ -0.5335 & 0.6597 & 37.4761 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0.3066 & 0.3388 & 3.4130 & 1.0000 & 0 \\ -0.4369 & 0.7668 & 18.4030 & 0 & 1.0000 \end{pmatrix}$$

and

$$z(t_0) = \begin{pmatrix} 19.1219 & 80.1483 & 1 & 0 & 0 \end{pmatrix}^T \quad (73)$$

yields the average state:

$$x_{av} = (\ 36.4284 \quad 71.5026\ )^T \quad (74)$$

$$x_{av,ss} = (\ 40.0000 \quad 80.0000\ )^T \quad (75)$$

The average value has also been calculated via state-space averaging. For this particular choice of parameters, the difference between the exact solution and the approximated value is about 10%.

The Fourier coefficients are found from (50) and (51) yielding

$$c_k = e^{A_{1,k}d_1T} \begin{pmatrix} x_0 & 1 & 0 & 0 \end{pmatrix}^T + \quad (76)$$

$$e^{A_{2,k}d_2T} \begin{pmatrix} x_1 & 1 & 0 & 0 \end{pmatrix}^T \quad (77)$$

The results for $k = 1, \ldots, 10$ have been listed in the table 1.

k	Four. coeff u_C	Four. coeff i_L
1	$-6.5609 + 2.7645j$	$4.1506 + 1.0473j$
2	$-0.2740 - 0.0215j$	$-0.4093 + 0.1082j$
3	$-0.7832 + 0.1024j$	$0.4066 + 0.0411j$
4	$-0.0646 - 0.0031j$	$-0.1036 + 0.0134j$
5	$-0.2835 + 0.0221j$	$0.1449 + 0.0089j$
6	$-0.0284 - 0.0009j$	$-0.0462 + 0.0040j$
7	$-0.1448 + 0.0081j$	$0.0737 + 0.0033j$
8	$-0.0159 - 0.0004j$	$-0.0260 + 0.0017j$
9	$-0.0877 + 0.0038j$	$0.0445 + 0.0015j$
10	$-0.0102 - 0.0002j$	$-0.0166 + 0.0009j$

Table 1: *Fourier coefficients*

The dynamics of the transient behaviour can also be calculated for both state-space averaging and the proposed method. Using (5) and (58), the eigenvalues are calculated:

$$\text{eigenvalues}(e^{A_{av}T}) = 0.3679 \pm 0.0000j \quad (78)$$

$$\text{eigenvalues}(e^{KT}) = 0.3531 \pm 0.1034j \quad (79)$$

Again, the differences between state-space averaging and the exact values are significant.

COMPARISON

Up to now, a number of different ways for calculating properties of the state-vector $x(t)$ of a system of a switching network have been described. If a system description as given by (1)-(2) is adopted, the system exhibits two types of behaviour:

- cyclic behaviour. The state-vector is a periodic function $x_{per}(t)$ with period T. This function is found by stating that $x_{per}(t+T) = x_{per}(t)$ for any t. If the system is in cyclic mode, several characteristic properties of the periodic state-vector can be calculated: average value, harmonics, extreme values.
- transient behaviour. If the system is not in cyclic mode, it will display a transient behaviour from $x(t_0)$ towards the cyclic mode, as described in (65). In controller design it is useful to know the dynamics with which $x(t)$ converges to the periodic $x_{per}(t)$.

In table 2 the two discussed analysis methods, state-space averaging and exact modelling, are mentioned with their capabilities to calculate $x_{per}(t)$, x_{av}, the Fourier coefficients and the extreme values of $x_{per}(t)$.

	$x_{per}(t)$	x_{av}	Four. coeff.	$\underline{x},\bar{x}$
State-space averaging	–	≈	–	–
Exact modelling	+	+	+	≈

Table 2: *Comparison methods in cyclic mode*

Clearly, the proposed method gives more information concerning the periodic state-vector $x_{per}(t)$.

With state-space averaging, the transient is fully determined by A_{av} and b_{av}, as illustrated in (5). However, (5) is only an approximation in which the dynamics are fully determined by the eigenvalues of A_{av}. The description of $x(t)$ via the extended Floquet theory, as elucidated in (65), yields an exact and appropriate way for analysing the transient behaviour of $x(t)$. Here, the dynamics are determined by the eigenvalues of the Floquet matrix K.

	dynamics	exact
State-space averaging	A_{av}	no
Floquet	K	yes

Table 3: *Comparison methods in transient mode*

In table (3) some characteristics of both methods are summarized. Again, state-space averaging yields only approximate results, while the Floquet expression (65) offers an accurate desription of the transient dynamics.

CONCLUSIONS

There are several ways to analyse switching systems. State-space averaging is mathematically simple, but offers only an approximation of the transient behaviour of the system and the average value of the state-vector $x(t)$ in steady-state. Two new exact modelling methods have been proposed. With these methods the average value, Fourier coefficients, extreme values and the transient behaviour of the system can be obtained analytically. The proposed methods are very well suited for accurate analysis of switched networks with fixed duty-cycles.

APPENDIX A

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} &= c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega t} + c_{-k} e^{-jk\omega t} \right) \\
&= c_0 + \sum_{k=1}^{\infty} \Big\{ c_k(\cos k\omega t + j \sin k\omega t) + \\
&\quad + c_{-k}(\cos(-k\omega t) + j \sin(-k\omega t)) \Big\} \\
&= c_0 + \sum_{k=1}^{\infty} \Big\{ (c_k + c_{-k}) \cos k\omega t + \\
&\quad + (c_k - c_{-k}) j \sin k\omega t \Big\} \\
&= c_0 + \sum_{k=1}^{\infty} (a_k \cos k\omega t + b_k \sin k\omega t)
\end{aligned}
$$

with $a_k = 2\mathrm{Re}(c_k)$ and $b_k = -2\mathrm{Im}(c_k)$.

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